Wehrli

Thm (Bar-Natan, W)

The homotopy type of Khovanov's chain complex with coeff in $\mathbb{Z}/2\mathbb{Z}$ is inv. under component preserving link mutation.

Definition of Bar-Natan's Khovanov bracket

L or link diagram $\longrightarrow$ resolution, cube in $Cob(SU_2)$ $\xrightarrow{\text{“cone”}}$ $Kh(L)$

$\cap$

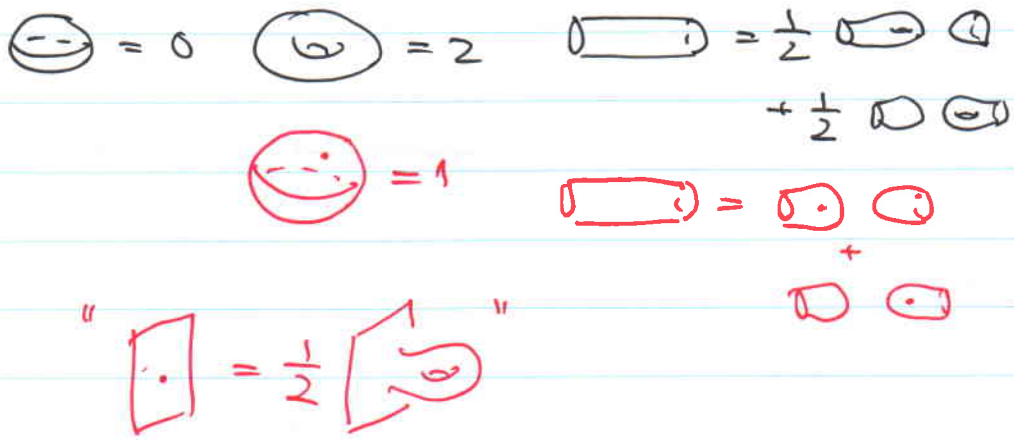
$Kom_{/h}(Mat(Cob(SU_2)))$

Category $Cob(SU_2)$

objects : unoriented flat diagrams

morphisms : formal $\cancel{\mathbb{C}}$ $\mathbb{Z}_2$-linear combinations of dotted-cobordisms mod relations

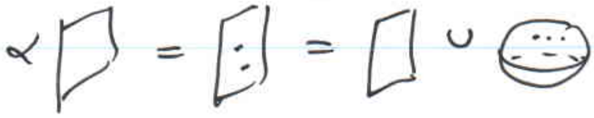
relations



Khovanov's setting $X \in \mathbb{Z}_2[X]/(X^2)$

Remarks

* morphism sets of $Cob(SU_2)$ are f.g. free graded modules over $\mathbb{Z}_2[\alpha]$ $\deg \alpha = -4$



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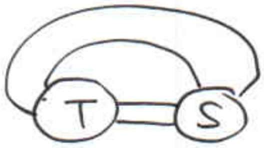
$\alpha = 0 \longrightarrow$ Khovanov

$\alpha = 1 \longrightarrow$ Lee

* T : (n,n)-tangle $\longrightarrow Kh(T)$ complex in $\mathrm{Mat}(\mathrm{Cob}(SU_2)(n \to n))$

* $Kh(\cdot)$ is a planar alg. morphism

L

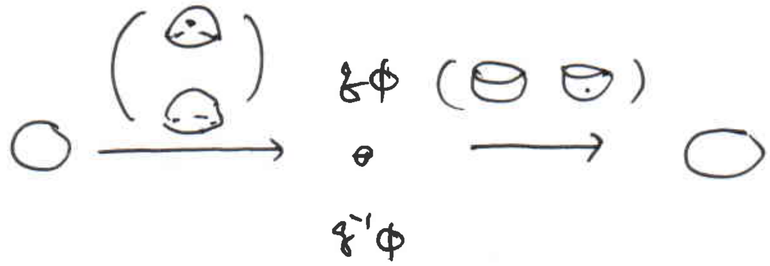


$Kh(L) = Kh(T) \otimes Kh(S)$

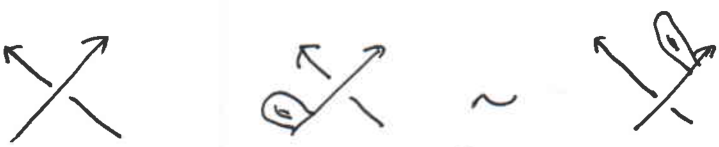
$d_L \qquad d_T \otimes I_S + I_T \otimes d_S$

<u>Delooping</u> in $\mathrm{Mat}(\mathrm{Cob}(SU_2))$

$\bigcirc \simeq q\phi + q^{-1}\phi$



Recall



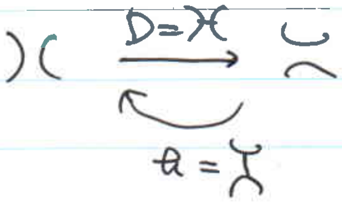
homotopic

We have similar :

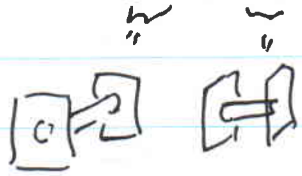
$X_1 \qquad X_2$

$X_1, X_2 : Kh(\times) \circlearrowleft$ homotopic

$Kh(\text{crossing})$:



$$Dh + hD = X_1 + X_2$$



I = identity of $Kh(\text{crossing})$

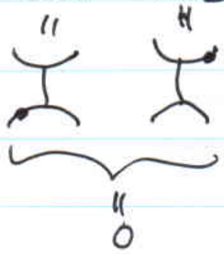
$$I + h : Kh(\text{crossing}) \circlearrowleft$$

$$\| \quad \begin{pmatrix} 1 & \text{saddle} \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & \text{saddle} \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2\,\text{saddle} \\ 0 & 1 \end{pmatrix} = I$$

$\therefore$ involution

$$(I+h)D(I+h) = D + \underbrace{Dh + hD}_{X_1 + X_2} + \underbrace{hDh}_{0}$$

$$\because \quad hDh = h(hD + \underbrace{[D,h]}_{X_1+X_2}) = \underbrace{h^2}_{0}D + hX_1 + hX_2$$



$$\therefore \quad (I+h)D(I+h) = D + X_1 + X_2$$

<u>Lemma</u> $\quad (Kh(\text{crossing}), D) \underset{I+h}{\cong} (Kh(\text{crossing}), D + X_1 + X_2)$

$$D + X_1 \xrightarrow{I+h} D + \cancel{X_1} + X_2 + \cancel{X_1} = D + X_2$$

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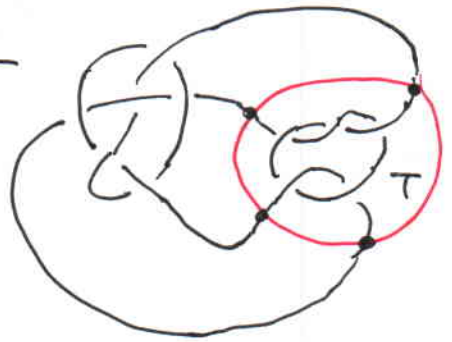
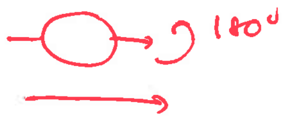
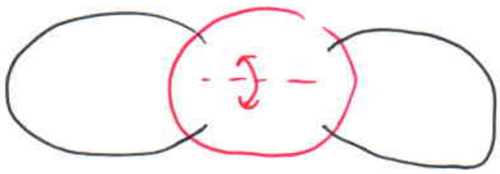
What is mutation?

L





special mutation



$\circlearrowright \xrightarrow{\text{after mutation}}$ in the same comp.

Exercise

Every component preserving mutation can be decomposed into a sequence of (3 dim) isotopies and special mutations

Th. Suppose L & L' are related by special mutation

$$\Longrightarrow Kh(L) \cong Kh(L')$$

ρ : flip $\qquad T' = \rho(T)$

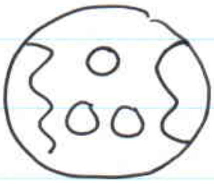
S : outside tangle (fixed)

Look at $Kh(T)$ & $Kh(T')$

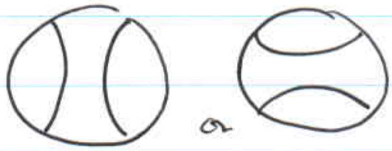
$$Kh(T)^k = \begin{matrix} O_1 \\ \oplus \\ \vdots \\ \oplus \\ O_m \end{matrix} \xrightarrow{\rho} Kh(T')^k = \begin{matrix} \rho O_1 \\ \oplus \\ \\ \oplus \\ \rho O_m \end{matrix}$$

$Kh(T), Kh(T')$ $\qquad\qquad$ $Kh(T)_{dl}$ $\quad$ $Kh(T')_{dl}$

resolutions of T

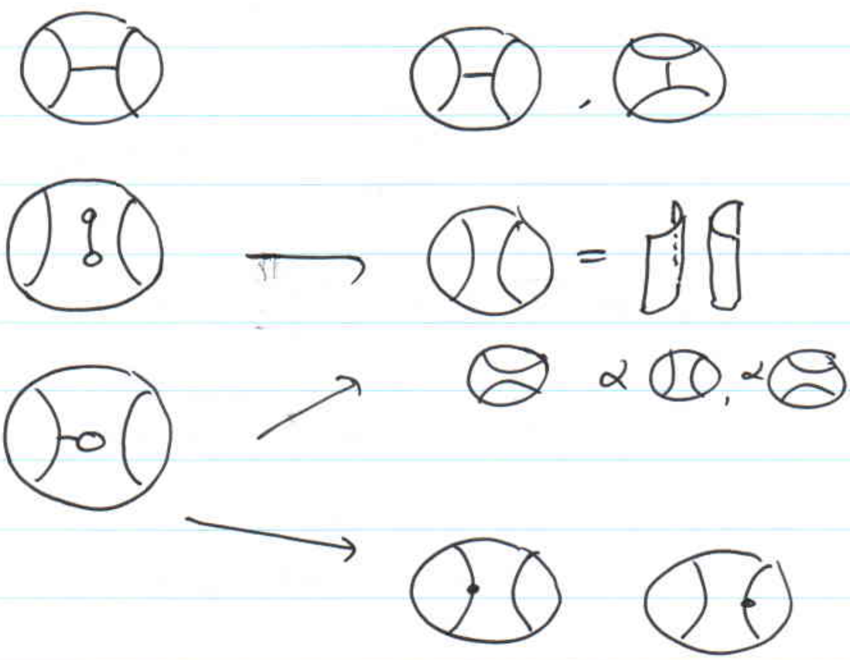


"delooping" all circles



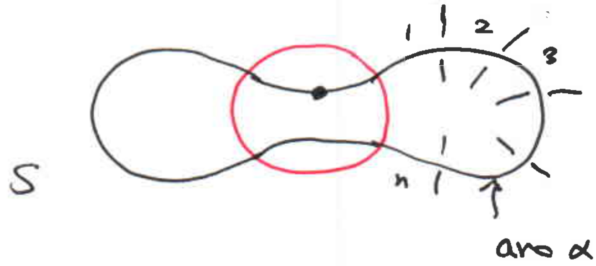
$$Kh(T)_{dl} = Kh(T')_{dl} \quad \text{on object level}$$

differentials $\quad Kh(T) \xrightarrow{\text{deloop}} Kh(T)_{dl}$



Only non-invariant

look at $Kh(S)$ $\qquad$ S = outside tangle



$X_i :=$ 

$X_i : Kh(S) \circlearrowleft$

$\exists$ homotopy h_i between X_i & X_{i+1}

$$Dh_i + h_i D = X_i + X_{i+1}$$

Lemma $\Rightarrow$ $\quad (I+h_i)\, D\, (I+h_i) = D + X_i + X_{i+1}$

$$D \xrightarrow[I+h_1]{\text{conj.}} D + X_1 + X_2 \xrightarrow[I+h_2]{\text{conj}} D + \cancel{X_2} + X_3 + X_1 + \cancel{X_2}$$

$$= D + X_1 + X_3$$

$$\to \cdots \xrightarrow[I+h_{n-1}]{} D + X_1 + X_n$$

$$\varphi := \prod (I + h_i) \; : \; Kh(S) \circlearrowleft \qquad \varphi^2 = I$$

$\shortparallel$

"$\exp(h_1 + \cdots + h_{n-1})$"

$$\varphi D \varphi = D + X_1 + X_n$$

$$D + X_1 \xrightarrow{\substack{\text{conj} \\ \text{w } \varphi}} D + X_n$$

$Kh(\text{link before mutation}) \cong Kh(T)_{de} \otimes Kh(S)$ $\qquad p \otimes I + I \otimes D$

$Kh(\text{link } \cancel{\text{after}} \text{ mut.}) \cong Kh(T')_{de} \otimes Kh(S)$ $\qquad g \otimes I + I \otimes D$

p = diff. of $Kh(T)$

g = diff of $Kh(T')$

D = diff of $Kh(S)$

$p = p_{nodot} + p_{dot}$

$$(p_{dot})_{ij} = \begin{cases} p_{ij} & \text{if } p_{ij} \text{ contains dot} \\ 0 & \text{otherwise} \end{cases}$$

$\tilde{p}$ = matrix obtained by removing all dots in p_{dot}

Example

$p_{ij} =$  $\quad (p_{dot})_{ij} =$  $\quad (p_{nodot})_{ij} = 0$

$\tilde{p}_{ij} =$ 

$g_{ij} =$ 

p_{ij}  $\otimes I =$  $\otimes X_1$

g_{ij}  $\otimes I =$  $\otimes X_2$

$\uparrow$

$\tilde{p}_{ij}$

$p_{ij} \otimes I + g_{ij} \otimes I$

$= \tilde{p}_{ij} \otimes (X_1 + X_2)$

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$$\Longrightarrow \quad (p+q)\otimes I = \tilde{p}\otimes(X_1+X_2)$$

$p_{nodot} = q_{nodot}$

$$\Phi := \prod_{i=1}^{n-1} (I\otimes I + \tilde{p}\otimes t_i)$$

① $\Phi(I\otimes D)\Phi = I\otimes D + \tilde{p}\otimes(X_1+X_n)$

$$\underset{\text{above}}{=} I\otimes D + (p+q)\otimes I$$

② $[p\otimes I, \tilde{p}\otimes t_i] = 0 \quad \forall i$

not obvious

$$\Rightarrow \quad [p\otimes I, \Phi] = 0$$

$$\therefore \quad p\otimes I + I\otimes D \xrightarrow[\text{w.r.t. } \Phi]{\text{conj.}} q\otimes I + I\otimes D$$